Exercise 4.1 Page No: 108

Evaluate the following determinants in Exercise 1 and 2.

1. $\begin{vmatrix} 2 & 4 \\ -5 & -1 \end{vmatrix}$

Solution:

$\begin{vmatrix} 2 & 4 \\ -5 & -1 \end{vmatrix}$ = 2(-1) – 4(-5) = 18

2.

(i) $\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$

(ii) $\begin{vmatrix} x^2 - x + 1 & x - 1 \\ x + 1 & x + 1 \end{vmatrix}$

Solution:

(i) $\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$ = cos θ x cos θ – (-sin θ) x sin θ = 1

(ii) $\begin{vmatrix} x^2 - x + 1 & x - 1 \\ x + 1 & x + 1 \end{vmatrix}$ = (x^2 – x + 1)(x + 1) – (x + 1)(x – 1) = x^3 – x^2 + 2

3. If A = $\begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix}$ **then show that |2A| = 4|A|.**

Solution:

A = $\begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix}$

2A = $\begin{bmatrix} 2 & 4 \\ 8 & 4 \end{bmatrix}$

L.H.S. = |2A| = $\begin{vmatrix} 2 & 4 \\ 8 & 4 \end{vmatrix}$ = 8 – 32 = -24

R.H.S. = 4|A| = 4$\begin{vmatrix} 1 & 2 \\ 4 & 2 \end{vmatrix}$ = 4 (2 – 8) = -24

LHS = RHS

4. If A = $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix}$ then show that | 3 A | = 27 | A |

Solution:

$$3A = \begin{bmatrix} 3 & 0 & 3 \\ 0 & 3 & 6 \\ 0 & 0 & 12 \end{bmatrix}$$

LHS:

$$|3A| = \begin{vmatrix} 3 & 0 & 3 \\ 0 & 3 & 6 \\ 0 & 0 & 12 \end{vmatrix}$$

= 3 x 36 = 108

RHS

$$27|A| = 27\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{vmatrix}$$

= 27(4) = 108

LHS = RHS

5. Evaluate the determinants

(i) $\begin{vmatrix} 3 & -1 & -2 \\ 0 & 0 & -1 \\ 3 & -5 & 0 \end{vmatrix}$ (ii) $\begin{vmatrix} 3 & -4 & 5 \\ 1 & 1 & -2 \\ 2 & 3 & 1 \end{vmatrix}$

(iii) $\begin{vmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ -2 & 3 & 0 \end{vmatrix}$ (iv) $\begin{vmatrix} 2 & -1 & -2 \\ 0 & 2 & -1 \\ 3 & -5 & 0 \end{vmatrix}$

Solution:

(i)

$$\begin{vmatrix} 3 & -1 & -2 \\ 0 & 0 & -1 \\ 3 & -5 & 0 \end{vmatrix} = 3\begin{vmatrix} 0 & -1 \\ -5 & 0 \end{vmatrix} - (-1)\begin{vmatrix} 0 & -1 \\ 3 & 0 \end{vmatrix} + (-2)\begin{vmatrix} 0 & 0 \\ 3 & -5 \end{vmatrix}$$

= -15 + 3 – 0 = -12

(ii)

$$\begin{vmatrix} 3 & -4 & 5 \\ 1 & 1 & -2 \\ 2 & 3 & 1 \end{vmatrix} = 3\begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix} - (-4)\begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} + 5\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix}$$

= 3(7) + 4(5) + 5(1)

= 46

(iii)

$$\begin{vmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ -2 & 3 & 0 \end{vmatrix} = 0\begin{vmatrix} 0 & -3 \\ 3 & 0 \end{vmatrix} - 1\begin{vmatrix} -1 & -3 \\ -2 & 0 \end{vmatrix} + 2\begin{vmatrix} -1 & 0 \\ -2 & 3 \end{vmatrix}$$

= 0 – 1(-6) + 2(-3-0)

= 0

(iv)

$$\begin{vmatrix} 2 & -1 & -2 \\ 0 & 2 & -1 \\ 3 & -5 & 0 \end{vmatrix} = 2\begin{vmatrix} 2 & -1 \\ -5 & 0 \end{vmatrix} - (-1)\begin{vmatrix} 0 & -1 \\ 3 & 0 \end{vmatrix} + (-2)\begin{vmatrix} 0 & 2 \\ 3 & -5 \end{vmatrix}$$

= 2(-5) + (0+3) – 2(0-6)

= 5

6. If A = $\begin{bmatrix} 1 & 1 & -2 \\ 2 & 1 & -3 \\ 5 & 4 & -9 \end{bmatrix}$ **find |A|.**

Solution:

$$|A| = \begin{vmatrix} 1 & 1 & -2 \\ 2 & 1 & -3 \\ 5 & 4 & -9 \end{vmatrix}$$

$$= 1\begin{vmatrix} 1 & -3 \\ 4 & -9 \end{vmatrix} - 1\begin{vmatrix} 2 & -3 \\ 5 & -9 \end{vmatrix} + (-2)\begin{vmatrix} 2 & 1 \\ 5 & 4 \end{vmatrix}$$

= 1(-9 + 12) – (-18 + 15) – 2(8 - 5)

= 0

7. Find values of x, if

(i) $\begin{vmatrix} 2 & 4 \\ 5 & 1 \end{vmatrix} = \begin{vmatrix} 2x & 4 \\ 6 & x \end{vmatrix}$ (ii) $\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = \begin{vmatrix} x & 3 \\ 2x & 5 \end{vmatrix}$

Solution:

(i)

$$\begin{vmatrix} 2 & 4 \\ 5 & 1 \end{vmatrix} = \begin{vmatrix} 2x & 4 \\ 6 & x \end{vmatrix}$$

2 – 20 = 2x^2 – 24

2x^2 = 6

x^2 = 3

or x = $\pm\sqrt{3}$

(ii)

$$\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = \begin{vmatrix} x & 3 \\ 2x & 5 \end{vmatrix}$$

10 – 12 = 5x – 6x
x = 2

8. If $\begin{vmatrix} x & 2 \\ 18 & x \end{vmatrix} = \begin{vmatrix} 6 & 2 \\ 18 & 6 \end{vmatrix}$ **then x is equal to**

(A) 6 **(B) ± 6** **(C) – 6** **(D) 0**

Solution:

Option (B) is correct.

Explanation:

$$\begin{vmatrix} x & 2 \\ 18 & x \end{vmatrix} = \begin{vmatrix} 6 & 2 \\ 18 & 6 \end{vmatrix}$$

x^2 -36 = 36 – 32

x^2 = 36

x = ±6

Exercise 4.2 Page No: 119

Using the property of determinants and without expanding in Exercises 1 to 7, prove that:

1. $\begin{vmatrix} x & a & x+a \\ y & b & y+b \\ z & c & z+c \end{vmatrix} = 0$

Solution:

L.H.S.

$$\begin{vmatrix} x & a & x+a \\ y & b & y+b \\ z & c & z+c \end{vmatrix}$$

Applying: $C_1 + C_2$

$$\begin{vmatrix} x+a & a & x+a \\ y+b & b & y+b \\ z+c & c & z+c \end{vmatrix}$$

Elements of Column 1 and Column 2 are same. So determinant value is zero as per determinant properties.

= 0

= RHS

Proved.

2. $\begin{vmatrix} a-b & b-c & a-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = 0$

Solution:

$$\begin{vmatrix} a-b & b-c & a-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$$

Applying: $C_1 \rightarrow C_1 + C_2 + C_3$

$$= \begin{vmatrix} 0 & b-c & a-a \\ 0 & c-a & a-b \\ 0 & a-b & b-c \end{vmatrix}$$

= 0

All entries of first column are zero. (As per determinant properties)

3. $\begin{vmatrix} 2 & 7 & 65 \\ 3 & 8 & 75 \\ 5 & 9 & 86 \end{vmatrix} = 0$

Solution:

$$\begin{vmatrix} 2 & 7 & 65 \\ 3 & 8 & 75 \\ 5 & 9 & 86 \end{vmatrix}$$

Applying: $C_3 \rightarrow C_3 - C_1$

$$\begin{vmatrix} 2 & 7 & 65 \\ 3 & 8 & 72 \\ 5 & 9 & 81 \end{vmatrix} = 9\begin{vmatrix} 2 & 7 & 7 \\ 3 & 8 & 8 \\ 5 & 9 & 9 \end{vmatrix}$$

Elements of 2 columns are same, so determinant is zero.

= 0
Proved.

4.
$$\begin{vmatrix} 1 & bc & a(b+c) \\ 1 & ca & b(c+a) \\ 1 & ab & c(a+b) \end{vmatrix} = 0$$

Solution:

$$\begin{vmatrix} 1 & bc & a(b+c) \\ 1 & ca & b(c+a) \\ 1 & ab & c(a+b) \end{vmatrix}$$

Applying: C_3 -> C_3 + C_2

$$\begin{vmatrix} 1 & bc & ab+ab+ac \\ 1 & ca & ab+ab+ac \\ 1 & ab & ab+ab+ac \end{vmatrix}$$

(ab + ab + ac) is a common element in 3rd row.

$$= (ab+ab+ac)\begin{vmatrix} 1 & bc & 1 \\ 1 & ca & 1 \\ 1 & ab & 1 \end{vmatrix}$$

Two columns are identical, so determinant is zero.

= 0

5. Prove that

$$\begin{vmatrix} (b+c) & q+r & y+z \\ (c+a) & r+p & z+x \\ (a+b) & p+q & x+y \end{vmatrix} = 2\begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix}$$

Solution:

LHS:

Applying: $R_1 \rightarrow R_1 + R_2 + R_3$

$$\begin{vmatrix} b+c+c+a+a+b & q+r+r+p+p+q & y+z+z+x+x+y \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix}$$

$$= 2\begin{vmatrix} (a+b+c) & (p+q+r) & (x+y+z) \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix}$$

Applying:

$R_1 \to R_1 - R_2$

and $R_3 \to R_3 - R_1$

$$2\begin{vmatrix} b & q & y \\ c+a & r+p & z+x \\ a & p & x \end{vmatrix}$$

Again, $R_2 \to R_2 - R_3$

$$2\begin{vmatrix} b & q & y \\ c & r & z \\ a & p & x \end{vmatrix}$$

Interchanging rows, we have

$$2\begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix}$$

= RHS

Proved.

6. Prove that

$$\begin{vmatrix} 0 & a & -b \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix} = 0$$

Solution:

Let $\Delta = \begin{vmatrix} 0 & a & -b \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix}$

Taking (-1) common from all the 3 rows. Again, interchanging rows and columns, we have

$$\Delta = -\begin{vmatrix} 0 & a & -b \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix}$$

$\Delta = -\Delta$

Which shows that, $2\Delta = 0$ or $\Delta = 0$. Proved.

7. Prove that

$$\begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ca & cb & -c^2 \end{vmatrix} = 4a^2b^2c^2$$

Solution: LHS:

$$\begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ca & cb & -c^2 \end{vmatrix}$$

Taking a, b, c from row 1, row and row 3 respectively,

$$= abc\begin{vmatrix} -a & a & a \\ a & -b & b \\ a & b & -c \end{vmatrix}$$

$$R_1 \rightarrow R_1 + R_2$$

$$= abc\begin{vmatrix} 0 & 0 & 2c \\ a & -b & c \\ a & b & -c \end{vmatrix}$$

$$= abc(2c)\begin{vmatrix} a & -b \\ a & b \end{vmatrix}$$

= 2abc^2 (ab + ab)

= $4a^2b^2c^2$

= RHS

Proved.

By using properties of determinants, in Exercises 8 to 14, show that:
8.

(i) $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$

(ii) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$

Solution:

(i)LHS:

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

$R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

$$= \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 1 & c-a & c^2-a^2 \end{vmatrix}$$

Expanding 1st column,

$$= 1\begin{vmatrix} b-a & b^2-a^2 \\ c-a & c^2-a^2 \end{vmatrix}$$

Taking (b-a) common from first row,

$$= (b-a)(c-a)\begin{vmatrix} 1 & b+a \\ 1 & c+a \end{vmatrix}$$

Simplifying above expression, we have

= (b-c)(c-a)(c-b)

= (a – b)(b – c)(c – a)

= RHS

Proved.

(ii) LHS

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix}$$

$C_2 \rightarrow C_2 - C_1$ and $C_3 \rightarrow C_3 - C_1$

$$= \begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^3 & b^3 - a^3 & c^3 - a^3 \end{vmatrix}$$

Expanding first row

$$= 1\begin{vmatrix} b-a & c-a \\ (b-a)(b^2+a^2+ab) & (c-a)(c^2+a^2+ac) \end{vmatrix}$$

$$= (b-a)(c-a)\begin{vmatrix} 1 & 1 \\ (b^2+a^2+ab) & (c^2+a^2+ac) \end{vmatrix}$$

$$= (b-a)(c-a)(c^2+a^2+ac-b^2-a^2-ab)$$

$$= (b-a)(c-a)(c^2-b^2+ac-ab)$$

$$= (b-a)(c-a)[(c-b)(c+b)+a(c-b)]$$

$$= (b-a)(c-a)(c-b)(c+b+a)$$

$$=(a-b)(b-c)(c-a)(a+b+c)$$

=RHS

Proved

9. Prove that

$$\begin{vmatrix} x & x^2 & yz \\ y & y^2 & zx \\ z & z^2 & xy \end{vmatrix} = (x-y)(y-z)(z-x)(xy+yz+zx)$$

Solution: LHS

$$\begin{vmatrix} x & x^2 & yz \\ y & y^2 & zx \\ z & z^2 & xy \end{vmatrix}$$

Mulitiplying R_1, R_2, R_3 by x, y, z respectively

$$\begin{vmatrix} x^2 & x^3 & xyz \\ y^2 & y^3 & xyz \\ z^2 & z^3 & xyz \end{vmatrix}$$

$$= \frac{xyz}{xyz}\begin{vmatrix} x^2 & x^3 & 1 \\ y^2 & y^3 & 1 \\ z^2 & z^3 & 1 \end{vmatrix} = \begin{vmatrix} x^2 & x^3 & 1 \\ y^2 & y^3 & 1 \\ z^2 & z^3 & 1 \end{vmatrix}$$

$R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$

$$= \begin{vmatrix} x^2 & x^3 & 1 \\ y^2 - x^2 & y^3 - x^3 & 0 \\ z^2 - x^2 & z^3 - x^3 & 0 \end{vmatrix}$$

$$= 1\begin{vmatrix} y^2 - x^2 & y^3 - x^3 \\ z^2 - x^2 & z^3 - x^3 \end{vmatrix}$$

$$= \begin{vmatrix} (y-x)(y+x) & (y-x)(y^2+x^2+yx) \\ (z-x)(z+x) & (z-x)(z^2+x^2+zx) \end{vmatrix}$$

$$= (y-x)(z-x)\begin{vmatrix} y+x & y^2+x^2+yx \\ z+x & z^2+x^2+zx \end{vmatrix}$$

$$= (y-x)(z-x)\left[yz^2 + yx^2 + xyz + xz^2 + x^3 + x^2z - zy^2 - zx^2 - xyz - xy^2 - x^3 - x^2y\right]$$

$$= (y-x)(z-x)\left[yz^2 - zy^2 + xz^2 - xy^2\right]$$

$$= (y-x)(z-x)\left[yz(z-y) + x(z^2 - y^2)\right]$$

$$= (y-x)(z-x)\left[yz(z-y) + x(z-y)(z+y)\right]$$

$$= (y-x)(z-x)(z-y)\left[yz + x(z+y)\right]$$

$$= (x-y)(y-z)(z-x)(xy+yz+zx)$$

RHS(Proved)

10.

(i) $\begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix} = (5x+4)(4-x)^2$

(ii) $\begin{vmatrix} y+k & y & y \\ y & y+k & y \\ y & y & y+k \end{vmatrix} = k^2(3y+k)$

Solution:
(i) LHS

$$\begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix}$$

$[R_1 \to R_1 + R_2 + R_3]$

$$= \begin{vmatrix} 5x+4 & 5x+4 & 5x+4 \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix}$$

$$= (5x+4)\begin{vmatrix} 1 & 1 & 1 \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix}$$

$C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$

$$= (5x+4)\begin{vmatrix} 1 & 0 & 0 \\ 2x & 4-x & 0 \\ 2x & 0 & 4-x \end{vmatrix}$$

$$= (5x+4).1\begin{vmatrix} 4-x & 0 \\ 0 & 4-x \end{vmatrix}$$

$$= (5x+4)(4-x)^2$$

= RHS (Proved)

(ii)LHS

$$\begin{vmatrix} y+k & y & y \\ y & y+k & y \\ y & 2x & y+k \end{vmatrix}$$

$C_1 \to C_1 + C_2 + C_3$

$$= \begin{vmatrix} 3y+k & y & y \\ 3y+k & y+k & y \\ 3y+k & y & y+k \end{vmatrix}$$

$$= (3y+k)\begin{vmatrix} 1 & y & y \\ 1 & y+k & y \\ 1 & y & y+k \end{vmatrix}$$

$C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$

$$= (3y+k)\begin{vmatrix} 1 & y & y \\ 0 & k & 0 \\ 0 & 0 & k \end{vmatrix}$$

$$= (3y+k).1\begin{vmatrix} k & 0 \\ 0 & k \end{vmatrix}$$

$$= k^2(3y+k)$$

RHS (Proved)

11. Prove that,

(i) $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$

(ii) $\begin{vmatrix} x+y+2z & x & y \\ z & y+z+2x & y \\ z & x & z+x+2y \end{vmatrix} = 2(x+y+z)^3$

Solution: LHS

$$\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

$R_1 \to R_1 + R_2 + R_3$

$$= \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

$$= (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

$C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$

$$= (a+b+c)\begin{vmatrix} 1 & 0 & 0 \\ 2b & -b-c-a & 0 \\ 2c & 0 & -c-a-b \end{vmatrix}$$

$$= (a+b+c)(1)\begin{vmatrix} -b-c-a & 0 \\ 0 & -c-a-b \end{vmatrix}$$

$$= (a+b+c)[-(b+c+a)][-(c+a+b)]$$

$$= (a+b+c)^3$$

(ii) LHS

$$\begin{vmatrix} x+y+2z & x & y \\ z & y+z+2x & y \\ z & x & z+x+2y \end{vmatrix}$$

$C_1 \to C_1 + C_2 + C_3$

$$= \begin{vmatrix} 2(x+y+z) & x & y \\ 2(x+y+z) & y+z+2x & y \\ 2(x+y+z) & x & z+x+2y \end{vmatrix}$$

Taking 2(x + y + z) common from first column. Then apply operations:

$R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

$$= 2(x+y+z)\begin{vmatrix} 1 & x & y \\ 0 & x+y+z & 0 \\ 0 & 0 & x+y+z \end{vmatrix}$$

$$=2(x+y+z)(1)\begin{vmatrix} x+y+z & 0 \\ 0 & x+y+z \end{vmatrix}$$

$$=2(x+y+z)\left[(x+y+z)^2-0\right]$$

= 2(x + y + z)^3
= RHS (Proved)

12. Prove that

$$\begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix} = (1-x^3)^2$$

Solution:
LHS

$$\begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix}$$

$R_1 \to R_1 + R_2 + R_3$

$$= \begin{vmatrix} 1+x+x^2 & 1+x+x^2 & 1+x+x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix}$$

$$= (1+x+x^2)\begin{vmatrix} 1 & 1 & 1 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix}$$

$C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$

$$= (1+x+x^2)\begin{vmatrix} 1 & 0 & 0 \\ x^2 & 1-x^2 & x-x^2 \\ x & x^2-x & 1-x \end{vmatrix}$$

$$= \left(1+x+x^2\right) \begin{vmatrix} 1-x^2 & x-x^2 \\ x^2-x & 1-x \end{vmatrix}$$

$$= \left(1+x+x^2\right) \begin{vmatrix} (1-x)(1+x) & x(1-x) \\ -x(1-x) & 1-x \end{vmatrix}$$

$$= \left(1+x+x^2\right)\left[(1-x)^2(1+x)+x^2(1-x)^2\right]$$

$$= \left(1+x+x^2\right)^2(1-x)^2$$

$$= \left(1-x+x-x^2+x^2-x^3\right)^2$$

$$= \left(1-x^3\right)^2$$

RHS
Proved.

13. Prove that

$$\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix} = \left(1+a^2+b^2\right)^3$$

Solution:
LHS

$$\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix}$$

$C_1 \to C_1 - b\,C_3$ and $C_2 \to C_2 + a\,C_3$

$$= \begin{vmatrix} 1+a^2+b^2 & 0 & -2b \\ 0 & 1+a^2+b^2 & 2a \\ b\left(1+a^2+b^2\right) & -a\left(1+a^2+b^2\right) & 1-a^2-b^2 \end{vmatrix}$$

$$= \left(1+a^2+b^2\right)^2 \begin{vmatrix} 1 & 0 & -2b \\ 0 & 1 & 2a \\ b & -a & 1-a^2-b^2 \end{vmatrix}$$

$R_3 \to R_3 - b\,R_1$

$$= \left(1+a^2+b^2\right)^2 \begin{vmatrix} 1 & 0 & -2b \\ 0 & 1 & 2a \\ 0 & -a & 1-a^2+b^2 \end{vmatrix}$$

$$= \left(1+a^2+b^2\right)^2 \begin{vmatrix} 1 & 2a \\ -a & 1-a^2+b^2 \end{vmatrix}$$

$$= \left(1+a^2+b^2\right)^2 \left(1-a^2+b^2+2a^2\right)$$

$$= \left(1+a^2+b^2\right)^3$$

RHS
Proved

14. Prove that

$$\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix} = 1+a^2+b^2+c^2$$

Solution: LHS

$$\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix}$$

Multiply, C_1, C_2, C_3 by a, b, c respectively

Then divide the determinant by abc

$$= \frac{1}{abc} \begin{vmatrix} a\left(a^2+1\right) & ab^2 & ac^2 \\ a^2b & b\left(b^2+1\right) & bc^2 \\ a^2c & b^2c & c\left(c^2+1\right) \end{vmatrix}$$

$$= \frac{abc}{abc} \begin{vmatrix} a^2+1 & b^2 & c^2 \\ a^2 & b^2+1 & c^2 \\ a^2 & b^2 & c^2+1 \end{vmatrix}$$

$C_1 \to C_1 + C_2 + C_3$

$$= \frac{abc}{abc}\begin{vmatrix} 1+a^2+b^2+c^2 & b^2 & c^2 \\ 1+a^2+b^2+c^2 & b^2+1 & c^2 \\ 1+a^2+b^2+c^2 & b^2 & c^2+1 \end{vmatrix}$$

$$= \left(1+a^2+b^2+c^2\right)\begin{vmatrix} 1 & b^2 & c^2 \\ 1 & b^2+1 & c^2 \\ 1 & b^2 & c^2+1 \end{vmatrix}$$

$R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$.

$$= \left(1+a^2+b^2+c^2\right)\begin{vmatrix} 1 & b^2 & c^2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \left(1+a^2+b^2+c^2\right)(1)(1-0)$$

$$= 1+a^2+b^2+c^2$$

LHS
(Proved)

Choose the correct answer in Exercises 15 and 16

15. Let A be a square matrix of order 3 × 3, then | kA| is equal to

(A) k | A | (B) k^2 | A | (C) k^3 | A | (D) 3k | A |

Solution:
Option (C) is correct.

16. Which of the following is correct
(A) Determinant is a square matrix.
(B) Determinant is a number associated to a matrix.
(C) Determinant is a number associated to a square matrix.
(D) None of these

Solution:
Option (C) is correct.

Exercise 4.3 Page No: 122

1. Find area of the triangle with vertices at the point given in each of the following:
(i) (1, 0), (6, 0), (4, 3)
(ii) (2, 7), (1, 1), (10, 8)
(iii) (–2, –3), (3, 2), (–1, –8)

Solution:

Formula for Area of triangle:

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

(i)

$$\frac{1}{2}\begin{vmatrix} 1 & 0 & 1 \\ 6 & 0 & 1 \\ 4 & 3 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\left[1(0-3)-0(6-4)+1(18-0)\right]$$

$$= \frac{15}{2} \text{ sq. units}$$

(ii)

$$\text{Area} = \frac{1}{2}\begin{vmatrix} 2 & 7 & 1 \\ 1 & 1 & 1 \\ 10 & 8 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\left[2(1-8)-7(1-10)+1(8-10)\right]$$

$$= \frac{47}{2} \text{ sq. unit}$$

(iii)

$$\text{Area} = \frac{1}{2}\begin{vmatrix} -2 & -3 & 1 \\ 3 & 2 & 1 \\ -1 & -8 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\left[-2(10)+3(4)-22\right]$$

= 15 sq. Units

2. Show that points: A (a, b + c), B (b, c + a), C (c, a + b) are collinear.

Solution:
Points are collinear if area of triangle is equal to zero.
i.e. Area of triangle = 0

$$\text{Area of Triangle} = \frac{1}{2}\begin{vmatrix} a & b+c & 1 \\ b & c+a & 1 \\ c & a+b & 1 \end{vmatrix}$$

$$= \frac{1}{2}\left[a(c+a-a-b)-(b+c)(b-c)+1\{b(a+b)-c(c+a)\}\right]$$

$$= \frac{1}{2}\left(ac-ab-b^2+c^2+ab+b^2-c^2-ac\right)$$

= 0
Therefore, points are collinear.

3. Find values of k if area of triangle is 4 sq. units and vertices are

(i) (k, 0), (4, 0), (0, 2)
(ii) (–2, 0), (0, 4), (0, k)

Solution:
(i)
Area of triangle = ±4 (Given)

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 4$$

$$\frac{1}{2}\left[k(0-2)-0+1(8-0)\right] = 4$$

½(-2k + 4) = 4

-k + 4 = 4
Now: -k + 4 = ±4
-k + 4 = 4 and –k + 4 = -4
k = 0 and k = 8

(ii) $\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 4$

$$\frac{1}{2}\begin{vmatrix} -2 & 0 & 1 \\ 0 & 4 & 1 \\ 0 & k & 1 \end{vmatrix} = 4$$

½(-8+2k) = 4

or -k + 4 = 4
Now: -k + 4 = ±4
-k + 4 = 4 and –k + 4 = -4
k = 0 and k = 8

4. (i) Find equation of line joining (1, 2) and (3, 6) using determinants.
(ii) Find equation of line joining (3, 1) and (9, 3) using determinants.

Solution:
Let A(x, y) be any vertex of a triangle.
All points are on one line if area of triangle is zero.

$$\frac{1}{2}\begin{vmatrix} 1 & 2 & 1 \\ 3 & 6 & 1 \\ x & y & 1 \end{vmatrix} = 0$$

$$\frac{1}{2}\left[x(2-6)-y(1-3)+1(6-6)\right]=0$$

-4x + 2y = 0

y = 2x

Which is equation of line.

(ii) Let A(x, y) be any vertex of a triangle.
All points are on one line if area of triangle is zero.

$$\frac{1}{2}\begin{vmatrix} x & y & 1 \\ 3 & 1 & 1 \\ 9 & 3 & 1 \end{vmatrix} = 0$$

$$\frac{1}{2}\left[x(1-3) - y(3-9) + 1(9-9)\right] = 0$$

-2x + 6y = 0

x – 3y = 0

Which is equation of line.

12. If area of triangle is 35 sq units with vertices (2, – 6), (5, 4) and (k, 4). Then k is

(A) 12 (B) –2 (C) –12, –2 (D) 12, –2

Solution:
Option (D) is correct.

Explanation:

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 35$$

$$\frac{1}{2}\left[2(4-4) - (-6)(5-k) + 1(20-4k)\right] = 35$$

Solving above expression, we have

25 – 5k = ±35

25 – 5k = 35 and 25 – 5k = -35

k = -2 and k = 12.

Exercise 4.4 Page No: 126

Write Minors and Cofactors of the elements of following determinants:

1.

(i) $\begin{vmatrix} 2 & -4 \\ 0 & 3 \end{vmatrix}$ (ii) $\begin{vmatrix} a & c \\ b & d \end{vmatrix}$

Solution:

Find Minors of elements:

Say, M_{ij} is minor of element a_{ij}

M_{11} = Minor of element a_{11} = 3

M_{12} = Minor of element a_{12} = 0

M_{21} = Minor of element a_{21} = -4

M_{22} = Minor of element a_{22} = 2

Find cofactor of a_{ij}

Let cofactor of a_{ij} is A_{ij}, which is $(-1)^{i+j} M_{ij}$

$A_{11} = (-1)^{1+1} M_{11} = (-1)^2 (3) = 3$

$A_{12} = (-1)^{1+2} M_{12} = (-1)^3 (0) = 0$

$A_{21} = (-1)^{2+1} M_{21} = (-1)^3 (-4) = 4$

$A_{22} = (-1)^{2+2} M_{22} = (-1)^4 (2) = 2$

(ii)

$\begin{vmatrix} a & c \\ b & d \end{vmatrix}$

Solution:

Find Minors of elements:

Say, M_{ij} is minor of element a_{ij}

M_{11} = Minor of element a_{11} = d

M_{12} = Minor of element a_{12} = b

M_{21} = Minor of element a_{21} = c

M_{22} = Minor of element a_{22} = a

Find cofactor of a_{ij}

Let cofactor of a_{ij} is A_{ij}, which is $(-1)^{i+j} M_{ij}$

$A_{11} = (-1)^{1+1} M_{11} = (-1)^2 (d) = d$

$A_{12} = (-1)^{1+2} M_{12} = (-1)^3 (b) = -b$

$A_{21} = (-1)^{2+1} M_{21} = (-1)^3 (c) = -c$

$A_{22} = (-1)^{2+2} M_{22} = (-1)^4 (a) = a$

2.

(i) $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$

(ii) $\begin{vmatrix} 1 & 0 & 4 \\ 3 & 5 & -5 \\ 0 & 2 & 2 \end{vmatrix}$

Solution:

(i) $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$

Find Minors and cofactors of elements:

Say, M_{ij} is minor of element a_{ij} and A_{ij} is cofactor of a_{ij}

M_{11} = Minor of element a_{11} = $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 - 0 = 1$ and $A_{11} = 1$

M_{12} = Minor of element a_{12} = $\begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0 - 0 = 0$ and $A_{12} = 0$

M_{13} = Minor of element a_{13} = $\begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 0 - 0 = 0$ and A_{13} =0

M_{21} = Minor of element a_{21} = $\begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0 - 0 = 0$ and A_{21} = 0

M_{22} = Minor of element a_{22} = $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 - 0 = 1$ and A_{22} = 1

M_{23} = Minor of element a_{23} = $\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0 - 0 = 0$ and A_{23} = 0

M_{31} = Minor of element a_{21} = $\begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} = 0 - 0 = 0$ and A_{31} = 0

M_{32} = Minor of element a_{32} = $\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0 - 0 = 0$ and A_{32} = 0

M_{33} = Minor of element a_{33} = $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 - 0 = 1$ and A_{33} = 1

(ii) $\begin{vmatrix} 1 & 0 & 4 \\ 3 & 5 & -5 \\ 0 & 2 & 2 \end{vmatrix}$

Find Minors and cofactors of elements:

Say, M_{ij} is minor of element a_{ij} and A_{ij} is cofactor of a_{ij}

M_{11} = Minor of element a_{11} = $\begin{vmatrix} 5 & -1 \\ 1 & 2 \end{vmatrix} = 10 - (-1) = 11$ and A_{11} = 11

M_{12} = Minor of element a_{12} = $\begin{vmatrix} 3 & -1 \\ 0 & 2 \end{vmatrix} = 6 - 0 = 6$ and A_{12} =-6

M_{13} = Minor of element a_{13} = $\begin{vmatrix} 3 & 5 \\ 0 & 1 \end{vmatrix} = 3 - 0 = 3$ and A_{13} =3

M_{21} = Minor of element a_{21} = $\begin{vmatrix} 0 & 4 \\ 1 & 2 \end{vmatrix} = 0 - 4 = -4$ and A_{21} = 4

M_{22} = Minor of element a_{22} = $\begin{vmatrix} 1 & 4 \\ 0 & 2 \end{vmatrix} = 2 - 0 = 2$ and A_{22} = 2

M_{23} = Minor of element a_{23} = $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 - 0 = 1$ and A_{23} = -1

M_{31} = Minor of element a_{31} = $\begin{vmatrix} 0 & 4 \\ 5 & -1 \end{vmatrix} = 0 - 20 = -20$ and A_{31} = -20

M_{32} = Minor of element a_{32} = $\begin{vmatrix} 1 & 4 \\ 3 & -1 \end{vmatrix} = -1 - 12 = -13$ and A_{32} = 13

M_{33} = Minor of element a_{33} = $\begin{vmatrix} 1 & 0 \\ 3 & 5 \end{vmatrix} = 5 - 0 = 5$ and A_{33} = 5

3. Using Cofactors of elements of second row, evaluate Δ.

$$\Delta = \begin{vmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}$$

Solution:
Find Cofactors of elements of second row:

A_{21} = Cofactor of element a_{21} = $(-1)^{2+1}\begin{vmatrix} 3 & 8 \\ 2 & 3 \end{vmatrix} = (-1)^3(9-16) = 7$

A_{22} = Cofactor of element a_{22} = $(-1)^{2+2}\begin{vmatrix} 5 & 8 \\ 1 & 3 \end{vmatrix} = (-1)^4(15-8) = 7$

A_{23} = Cofactor of element a_{23} = $(-1)^{2+3}\begin{vmatrix} 5 & 3 \\ 1 & 2 \end{vmatrix} = (-1)^5(10-3) = -7$

Now, $\Delta = a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23}$ = 14 + 0 – 7 = 7

4. Using Cofactors of elements of third column, evaluate Δ.

$$\Delta = \begin{vmatrix} 1 & x & yz \\ 1 & y & zx \\ 1 & z & xy \end{vmatrix}$$

Solution:

Find Cofactors of elements of third column:

A_{13} = Cofactor of element a_{13} = $(-1)^{1+3}\begin{vmatrix} 1 & y \\ 1 & z \end{vmatrix} = (-1)^4(z-y) = z-y$

A_{23} = Cofactor of element a_{23} = $(-1)^{2+3}\begin{vmatrix} 1 & x \\ 1 & z \end{vmatrix} = (-1)^5(z-x) = x-z$

A_{33} = Cofactor of element a_{33} = $(-1)^{3+3}\begin{vmatrix} 1 & x \\ 1 & y \end{vmatrix} = (-1)^6(y-x) = y-x$

Now, $\Delta = a_{13}A_{13} + a_{23}A_{23} + a_{33}A_{33}$

$= yz(z-y)+zx(x-z)+xy(y-x)$

$= (yz^2-y^2z)+(xy^2-xz^2)+(xz^2-x^2y)$

$= (y-z)[-yz+x(y+z)-x^2]$

$=(y-z)[-y(z-x)+x(z-x)]$

= (x – y)(y – x)(z – x)

5. If $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ **and A_{ij} is cofactor of a_{ij} then value of Δ is given by:**

(A) $a_{11}A_{31}+a_{12}A_{32}+a_{13}A_{33}$

(B) $a_{11}A_{11}+a_{12}A_{21}+a_{13}A_{31}$

(C) $a_{21}A_{11}+a_{22}A_{12}+a_{23}A_{13}$

(D) $a_{11}A_{11}+a_{21}A_{21}+a_{31}A_{31}$

Solution: Option (D) is correct.

Exercise 4.5 Page No: 131

Find adjoint of each of the matrices in Exercises 1 and 2.

1.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

2.

$$\begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 5 \\ -2 & 0 & 1 \end{bmatrix}$$

Solution:

1. Let A =

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Cofactors of the above matrix are

$A_{11} = 4$

$A_{12} = -3$

$A_{21} = -2$

$A_{22} = 1$

$$\text{adj. } A = \begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

2.

$$\text{Let } A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 5 \\ -2 & 0 & 1 \end{bmatrix}$$

Cofactors of the above matrix are

$$A_{11} = +\begin{vmatrix} 3 & 5 \\ 0 & 1 \end{vmatrix} = 3 \qquad A_{21} = -\begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} = 1 \qquad A_{31} = +\begin{vmatrix} -1 & 2 \\ 3 & 5 \end{vmatrix} = -11$$

$$A_{12} = -\begin{vmatrix} 2 & 5 \\ -2 & 1 \end{vmatrix} = -12 \qquad A_{22} = +\begin{vmatrix} 1 & 2 \\ -2 & 1 \end{vmatrix} = 5 \qquad A_{32} = -\begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} = -1$$

$$A_{13} = +\begin{vmatrix} 2 & 3 \\ -2 & 0 \end{vmatrix} = 6 \qquad A_{23} = -\begin{vmatrix} 1 & -1 \\ -2 & 0 \end{vmatrix} = 2 \qquad A_{33} = +\begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} = 5$$

Therefore,

$$\text{adj. A} = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = \begin{bmatrix} 3 & 1 & -11 \\ -12 & 5 & -1 \\ 6 & 2 & 5 \end{bmatrix}$$

Verify A (adj A) = (adj A) A = |A| I in Exercises 3 and 4

3. $\begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix}$

4. $\begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{bmatrix}$

Solution:

3.

$$\text{Let A} = \begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix}$$

$$\text{adj. A} = \begin{bmatrix} -6 & -3 \\ 4 & 2 \end{bmatrix}$$

$$\text{A.(adj. A)} = \begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix}\begin{bmatrix} -6 & -3 \\ 4 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Again, $|A| = \begin{vmatrix} 2 & 3 \\ -4 & -6 \end{vmatrix} = -12 + 12 = 0$

$$|A| I = (0)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

LHS = RHS

Verified.

4.

$$\text{Let A} = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{bmatrix}$$

Cofactors of A,

$A_{11} = +\begin{vmatrix} 0 & -2 \\ 0 & 3 \end{vmatrix} = 0 \qquad A_{21} = -\begin{vmatrix} -1 & 2 \\ 0 & 3 \end{vmatrix} = 3 \qquad A_{31} = +\begin{vmatrix} -1 & 2 \\ 0 & -2 \end{vmatrix} = 2$

$A_{12} = -\begin{vmatrix} 3 & -2 \\ 1 & 3 \end{vmatrix} = -11 \qquad A_{22} = +\begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} = 1 \qquad A_{32} = -\begin{vmatrix} 1 & 2 \\ 3 & -2 \end{vmatrix} = 8$

$A_{13} = +\begin{vmatrix} 3 & 0 \\ 1 & 0 \end{vmatrix} = 0 \qquad A_{23} = -\begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} = -1 \qquad A_{33} = +\begin{vmatrix} 1 & -1 \\ 3 & 0 \end{vmatrix} = 3$

Now,

$$\text{adj. } A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = \begin{bmatrix} 0 & 3 & 2 \\ -11 & 1 & 8 \\ 0 & -1 & 3 \end{bmatrix}$$

Now, Verify A (adj A) = (adj A) A = |A| I

$$A(\text{adj. } A) = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{bmatrix}\begin{bmatrix} 0 & 3 & 2 \\ -11 & 1 & 8 \\ 0 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix}$$

$$(\text{adj. } A)A = \begin{bmatrix} 0 & 3 & 2 \\ -11 & 1 & 8 \\ 0 & -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{vmatrix} = 1(0) - (-1)(11) + 2(0) = 11$$

$$|A|I = 11\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix}$$

Verified.

Find the inverse of each of the matrices (if it exists) given in Exercises 5 to 11.

5.

$$\begin{bmatrix} 2 & -2 \\ 4 & 3 \end{bmatrix}$$

Solution:

$$\text{Let } A = \begin{bmatrix} 2 & -2 \\ 4 & 3 \end{bmatrix}$$

$$|A| = \begin{bmatrix} 2 & -2 \\ 4 & 3 \end{bmatrix} = 14 \neq 0$$

Since determinant of the matrix is not zero, so inverse of this matrix is possible.

As we know, formula to find matrix inverse is:

$$A^{-1} = \frac{1}{|A|} \text{adj.} A$$

$$\text{adj. } A = \begin{bmatrix} 3 & 2 \\ -4 & 2 \end{bmatrix}$$

This implies,

$$A^{-1} = \frac{1}{14} \begin{bmatrix} 3 & 2 \\ -4 & 2 \end{bmatrix}$$

6.

$$\begin{bmatrix} -1 & 5 \\ -3 & 2 \end{bmatrix}$$

Solution:

$$\text{Let } A = \begin{bmatrix} -1 & 5 \\ -3 & 2 \end{bmatrix}$$

$$|A| = \begin{bmatrix} -1 & 5 \\ -3 & 2 \end{bmatrix} = 13 \neq 0$$

Since determinant of the matrix is not zero, so inverse of this matrix is possible.

As we know, formula to find matrix inverse is:

$$A^{-1} = \frac{1}{|A|} \text{adj.} A$$

$$\text{adj. A} = \begin{bmatrix} 2 & -5 \\ 3 & -1 \end{bmatrix}$$

This implies,

$$A^{-1} = \frac{1}{13}\begin{bmatrix} 2 & -5 \\ 3 & -1 \end{bmatrix}$$

7.

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}$$

Solution:

$$\text{Let } A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{vmatrix} = 1(10) - 2(0) + 3(0) = 10 \neq 0$$

Therefore,

A^{-1} exists

Find adj A:

$$A_{11} = +\begin{vmatrix} 2 & 4 \\ 0 & 5 \end{vmatrix} = 10 \quad A_{21} = -\begin{vmatrix} 2 & 3 \\ 0 & 5 \end{vmatrix} = -10 \quad A_{31} = +\begin{vmatrix} 2 & 3 \\ 2 & 4 \end{vmatrix} = 2$$

$$A_{12} = -\begin{vmatrix} 0 & 4 \\ 0 & 5 \end{vmatrix} = 0 \quad A_{22} = +\begin{vmatrix} 1 & 3 \\ 0 & 5 \end{vmatrix} = 5 \quad A_{32} = -\begin{vmatrix} 1 & 3 \\ 0 & 4 \end{vmatrix} = -4$$

$$A_{13} = +\begin{vmatrix} 0 & 2 \\ 0 & 0 \end{vmatrix} = 0 \quad A_{23} = -\begin{vmatrix} 1 & 2 \\ 0 & 0 \end{vmatrix} = 0 \quad A_{33} = +\begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix} = 2$$

As we know, formula to find matrix inverse is:

$$A^{-1} = \frac{1}{|A|}\text{adj.A}$$

$$A^{-1} = \frac{1}{10}\begin{bmatrix} 10 & -10 & 2 \\ 0 & 5 & -4 \\ 0 & 0 & 2 \end{bmatrix}$$

8.

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 3 & 0 \\ 5 & 2 & -1 \end{bmatrix}$$

Solution:

Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 3 & 0 \\ 5 & 2 & -1 \end{bmatrix}$

$$|A| = \begin{vmatrix} 1 & 0 & 0 \\ 3 & 3 & 0 \\ 5 & 2 & -1 \end{vmatrix} = 1(-3) - 0 + 0 = -3 \neq 0$$

Therefore,

A^{-1} exists

Find adj A:

$$A_{11} = +\begin{vmatrix} 3 & 0 \\ 2 & -1 \end{vmatrix} = -3 \quad A_{21} = -\begin{vmatrix} 0 & 0 \\ 2 & -1 \end{vmatrix} = 0 \quad A_{31} = +\begin{vmatrix} 0 & 0 \\ 3 & 0 \end{vmatrix} = 0$$

$$A_{12} = -\begin{vmatrix} 3 & 0 \\ 5 & -1 \end{vmatrix} = 3 \quad A_{22} = +\begin{vmatrix} 1 & 0 \\ 5 & -1 \end{vmatrix} = -1 \quad A_{32} = -\begin{vmatrix} 1 & 0 \\ 3 & 0 \end{vmatrix} = 0$$

$$A_{13} = +\begin{vmatrix} 3 & 3 \\ 5 & 2 \end{vmatrix} = -9 \quad A_{23} = -\begin{vmatrix} 1 & 0 \\ 5 & 2 \end{vmatrix} = -2 \quad A_{33} = +\begin{vmatrix} 1 & 0 \\ 3 & 3 \end{vmatrix} = 3$$

$$\text{adj. } A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = \begin{bmatrix} -3 & 0 & 0 \\ 3 & -1 & 0 \\ -9 & -2 & 3 \end{bmatrix}$$

As we know, formula to find matrix inverse is:

$$A^{-1} = \frac{1}{|A|} adj.A$$

$$A^{-1} = \frac{-1}{3}\begin{bmatrix} -3 & 0 & 0 \\ 3 & -1 & 0 \\ -9 & -2 & 3 \end{bmatrix}$$

9.

$$\begin{bmatrix} 2 & 1 & 3 \\ 4 & -1 & 0 \\ -7 & 2 & 1 \end{bmatrix}$$

Solution:

$$\text{Let } A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & -1 & 0 \\ -7 & 2 & 1 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 2 & 1 & 3 \\ 4 & -1 & 0 \\ -7 & 2 & 1 \end{vmatrix}$$

$=2(-1)-1(4) + 3(1) = -3$

$\neq 0$

Therefore,

A^{-1} exists

Find adj A:

$A_{11} = +\begin{vmatrix} -1 & 0 \\ 2 & 1 \end{vmatrix} = -1$ $A_{21} = -\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} = 5$ $A_{31} = +\begin{vmatrix} 1 & 3 \\ -1 & 0 \end{vmatrix} = 3$

$A_{12} = -\begin{vmatrix} 4 & 0 \\ -7 & 1 \end{vmatrix} = -4$ $A_{22} = +\begin{vmatrix} 2 & 3 \\ -7 & 1 \end{vmatrix} = 23$ $A_{32} = -\begin{vmatrix} 2 & 3 \\ 4 & 0 \end{vmatrix} = 12$

$A_{13} = +\begin{vmatrix} 4 & -1 \\ -7 & 2 \end{vmatrix} = 1$ $A_{23} = -\begin{vmatrix} 2 & 1 \\ -7 & 2 \end{vmatrix} = -11$ $A_{33} = +\begin{vmatrix} 2 & 1 \\ 4 & -1 \end{vmatrix} = -6$

$$\text{adj. } A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = \begin{bmatrix} -1 & 5 & 3 \\ -4 & 23 & 12 \\ 1 & -11 & -6 \end{bmatrix}$$

As we know, formula to find matrix inverse is:

$$A^{-1} = \frac{1}{|A|}\text{adj.}A$$

$$A^{-1} = \frac{-1}{3}\begin{bmatrix} -1 & 5 & 3 \\ -4 & 23 & 12 \\ 1 & -11 & -6 \end{bmatrix}$$

10.

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}$$

Solution:

$$\text{Let } A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{vmatrix}$$

$= 1(2)+1(9)+2(-6) = -1$

$\neq 0$

Therefore,

A^{-1} exists

Find adj A:

$A_{11} = +\begin{vmatrix} 2 & -3 \\ -2 & 4 \end{vmatrix} = 2$ $A_{21} = -\begin{vmatrix} -1 & 2 \\ -2 & 4 \end{vmatrix} = 0$ $A_{31} = +\begin{vmatrix} -1 & 2 \\ 2 & -3 \end{vmatrix} = -1$

$A_{12} = -\begin{vmatrix} 0 & -3 \\ 3 & 4 \end{vmatrix} = -9$ $A_{22} = +\begin{vmatrix} 2 & 2 \\ 3 & 4 \end{vmatrix} = -2$ $A_{32} = -\begin{vmatrix} 1 & 2 \\ 0 & -3 \end{vmatrix} = 3$

$A_{13} = +\begin{vmatrix} 0 & 2 \\ 3 & -2 \end{vmatrix} = -6$ $A_{23} = -\begin{vmatrix} 1 & -1 \\ 3 & -2 \end{vmatrix} = -1$ $A_{33} = +\begin{vmatrix} 1 & -1 \\ 0 & 2 \end{vmatrix} = 2$

$$\text{adj. } A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = \begin{bmatrix} 2 & 0 & -1 \\ -9 & -2 & 3 \\ -6 & -1 & 2 \end{bmatrix}$$

As we know, formula to find matrix inverse is:

$$A^{-1} = \frac{1}{|A|}\text{adj.}A$$

$$A^{-1} = \begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}$$

11.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & \sin\alpha & -\cos\alpha \end{bmatrix}$$

Solution:

$$\text{Let } A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & \sin\alpha & -\cos\alpha \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & \sin\alpha & -\cos\alpha \end{vmatrix}$$

$$= \left(-\cos^2\alpha - \sin^2\alpha\right) - 0 + 0$$

$$= -\left(\cos^2\alpha + \sin^2\alpha\right) = -1 \neq 0$$

Therefore,

A^{-1} exists

Find adj A:

$A_{11} = +\begin{vmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{vmatrix} = -1 \quad A_{21} = -\begin{vmatrix} 0 & 0 \\ \sin\alpha & -\cos\alpha \end{vmatrix} = 0 \quad A_{31} = +\begin{vmatrix} 0 & 0 \\ \cos\alpha & \sin\alpha \end{vmatrix} = 0$

$A_{12} = -\begin{vmatrix} 0 & \sin\alpha \\ 0 & -\cos\alpha \end{vmatrix} = 0 \quad A_{22} = +\begin{vmatrix} 1 & 0 \\ 0 & -\cos\alpha \end{vmatrix} = -\cos\alpha \quad A_{32} = -\begin{vmatrix} 1 & 0 \\ 0 & \sin\alpha \end{vmatrix} = -\sin\alpha$

$A_{13} = +\begin{vmatrix} 0 & \cos\alpha \\ 0 & \sin\alpha \end{vmatrix} = 0 \quad A_{23} = -\begin{vmatrix} 1 & 0 \\ 0 & \sin\alpha \end{vmatrix} = \sin\alpha \quad A_{33} = +\begin{vmatrix} 1 & 0 \\ 0 & \cos\alpha \end{vmatrix} = \cos\alpha$

$$\text{adj. } A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -\cos\alpha & -\sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix}$$

As we know, formula to find matrix inverse is:

$$A^{-1} = \frac{1}{|A|}\text{adj.}A$$

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & \sin\alpha & -\cos\alpha \end{bmatrix}$$

12. Let A = $\begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}$ **and B =** $\begin{bmatrix} 6 & 8 \\ 7 & 9 \end{bmatrix}$ **verify that** $(AB)^{-1} = B^{-1}A^{-1}$ **.**

Solution:

$$A = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 3 & 7 \\ 2 & 5 \end{vmatrix} = 1 \neq 0$$

$$A^{-1} = \frac{1}{|A|}\text{ adj. } A$$

$$A^{-1} = \begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix}$$

Again,

$$B = \begin{bmatrix} 6 & 8 \\ 7 & 9 \end{bmatrix}$$

$$|B| = \begin{vmatrix} 6 & 8 \\ 7 & 9 \end{vmatrix} = -2 \neq 0$$

$$B^{-1} = \frac{1}{-2}\begin{bmatrix} 9 & -8 \\ -7 & 6 \end{bmatrix}$$

Now Multiply A and B,

$$AB = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 6 & 8 \\ 7 & 9 \end{bmatrix} = \begin{bmatrix} 67 & 87 \\ 47 & 61 \end{bmatrix}$$

Find determinant of AB:

$$|AB| = \begin{vmatrix} 67 & 87 \\ 47 & 61 \end{vmatrix} = 4087 - 4089 = -2 \neq 0$$

Now, Verify $(AB)^{-1} = B^{-1} A^{-1}$

LHS:

$$|AB| = \begin{vmatrix} 67 & 87 \\ 47 & 61 \end{vmatrix} = 4087 - 4089 = -2 \neq 0$$

and

$$(AB)^{-1} = \frac{1}{-2}\begin{bmatrix} 61 & -87 \\ -47 & 67 \end{bmatrix}$$

RHS:

$$B^{-1}A^{-1} = \frac{1}{-2}\begin{bmatrix} 9 & -8 \\ -7 & 6 \end{bmatrix}\begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix}$$

$$= \frac{1}{-2}\begin{bmatrix} 61 & -87 \\ -47 & 67 \end{bmatrix}$$

This implies, LHS = RHS (Verified)

13. If A = $\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, show that $A^2 - 5A + 7I = O$. Hence find A^{-1}.

Solution:
$A^2 = AA$

$$A^2 = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}$$

LHS = $A^2 - 5A + 7I$

$$= \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - 5\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8-15+7 & 5-5+0 \\ -5+5+0 & 3-10+7 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

= RHS. (Proved)

To Find A^{-1}

Multiply $A^2 - 5A + 7I$ by A^{-1}, we have
(Consider I is 2x2 matrix)

$$A^2A^{-1} - 5A.A^{-1} + 7I.A^{-1} = 0.A^{-1}$$

$$A - 5I + 7A^{-1} = 0$$

$$7A^{-1} = -A + 5I$$

$$= \begin{bmatrix} -3 & -1 \\ 1 & -2 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{7}\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$$

14. For the matrix A = $\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$, find the numbers a and b such that $A^2 + aA + bI = O$.

Solution:

$$A^2 = AA = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 11 & 8 \\ 4 & 3 \end{bmatrix}$$

Since $A^2 + aA + bI = O$

$$\begin{bmatrix} 11 & 8 \\ 4 & 3 \end{bmatrix} + a\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} + b\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 0$$

$$\begin{bmatrix} 11 & 8 \\ 4 & 3 \end{bmatrix} + \begin{bmatrix} 3a & 2a \\ a & a \end{bmatrix} + \begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 11+3a+b & 8+2a+0 \\ 4+a+0 & 3+a+b \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Equate corresponding elements, we get

11 + 3a + b = 0 …(1)
8 + 2a = 0 => a = -4

Substitute the value of a in equation (1),
11 + 3(-4) + b = 0
11 – 12 + b = 0
b = 1.

15. For the matrix A = $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$. Show that $A^3 - 6A^2 + 5A + 11I = O$. Hence, find A^{-1}.

Solution:

$$A^2 = AA = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1+1+1 & 1+2-1 & 1-3+3 \\ 1+2-6 & 1+4+3 & 1-6-9 \\ 2-1+6 & 2-2-3 & 2+3+9 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix}$$

$$A^3 = A^2 A = \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 4+2+2 & 4+4-1 & 4-6+3 \\ -3+8-28 & -3+16+14 & -3-24-42 \\ 7-3+28 & 7-6-14 & 7+9+42 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 7 & 1 \\ -23 & 27 & -69 \\ 32 & -13 & 58 \end{bmatrix}$$

Now, LHS = $A^3 - 6A^2 + 5A + 11\,I$

$$= \begin{bmatrix} 8 & 7 & 1 \\ -23 & 27 & -69 \\ 32 & -13 & 58 \end{bmatrix} - \begin{bmatrix} 24 & 12 & 6 \\ -18 & 48 & -84 \\ 42 & -18 & 84 \end{bmatrix} + \begin{bmatrix} 5 & 5 & 5 \\ 5 & 10 & -15 \\ 10 & -5 & 15 \end{bmatrix} + \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix}$$

$$= \begin{bmatrix} 8-24+5+11 & 7-12+5 & 1-6+5 \\ -23+18+5 & 27-48+10+11 & -69+84-15 \\ 32-42+10 & -13+18-5 & 58-84+15+11 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

= 0

RHS (Proved)

Now, **find A^{-1}**

Multiply $A^3 - 6A^2 + 5A + 11\,I$ by A^{-1} , we have
(Consider I is 3x3 matrix)

$$A^3A^{-1} - 6A^2A^{-1} + 5AA^{-1} + 11I.A^{-1} = 0\,A^{-1}$$

$$A^2 - 6A + 5I + 11A^{-1} = 0$$

$$11A^{-1} = 6A - 5I - A^2$$

$$11A^{-1} = 6\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix}$$

$$= \begin{bmatrix} 6-5-4 & 6-2 & 6-1 \\ 6+3 & 12-5-8 & -18+14 \\ 12-7 & -6+3 & 18-5-14 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & 4 & 5 \\ 9 & -1 & -4 \\ 5 & -3 & -1 \end{bmatrix}$$

Therefore,

$$A^{-1} = \frac{1}{11}\begin{bmatrix} -3 & 4 & 5 \\ 9 & -1 & -4 \\ 5 & -3 & -1 \end{bmatrix}$$

16. If A = $\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$. Verify that $A^3 - 6A^2 + 9A - 4I = O$ and hence find A^{-1} .

Solution:

$A^2 = AA$

$$\begin{bmatrix} 4+1+1 & -2-2-1 & 1+1+2 \\ -2-2-1 & 1+4+1 & -1-2-2 \\ 2+1+2 & -1-2-2 & 1+1+4 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

Again, $A^3 = A^2A$

$$\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

Now, $\mathbf{A^3 - 6A^2 + 9A - 4I}$

$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - 6\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} + 9\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 22-36 & -21+30 & 21-30 \\ -21+30 & 22-36 & -21+30 \\ 21-30 & -21+30 & 22-36 \end{bmatrix} + \begin{bmatrix} 18-4 & -9-0 & 9-0 \\ -9-0 & 18-4 & -9-0 \\ 9-0 & -9-0 & 18-4 \end{bmatrix}$$

$$= \begin{bmatrix} -14+14 & 9-9 & -9+9 \\ 9-9 & -14+14 & 9-9 \\ -9+9 & 9-9 & -14+14 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

= 0 (RHS)

Multiply $A^3 - 6A^2 + 9A - 4I = O$ by A^{-1}, (here I is 3x3 matrix)

$$A^3A^{-1} - 6A^2A^{-1} + 9AA^{-1} - 4I.A^{-1} = 0.A^{-1}$$

$$A^2 - 6A + 9I - 4A^{-1} = 0$$

$$4A^{-1} = A^2 - 6A + 9I$$

Now Placing all the matrices,

$$4A^{-1}=\begin{bmatrix}6 & -5 & 5\\ -5 & 6 & -5\\ 5 & -5 & 6\end{bmatrix}-6\begin{bmatrix}2 & -1 & 1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}+9\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$4A^{-1}=\begin{bmatrix}6-12+9 & -5+6+0 & 5-6+0\\ -5+6+0 & 6-12+9 & -5+6+0\\ 5-6+0 & -5+6+0 & 6-12+9\end{bmatrix}$$

$$=\begin{bmatrix}3 & 1 & -1\\ 1 & 3 & 1\\ -1 & 1 & 3\end{bmatrix}$$

Inverse of the matrix is :

$$A^{-1}=\frac{1}{4}\begin{bmatrix}3 & 1 & -1\\ 1 & 3 & 1\\ -1 & 1 & 3\end{bmatrix}$$

17. Let A be a non-singular matrix of order 3 x 3. Then |adj. A| is equal to:
(A) |A| (B) $|A|^2$ (C) $|A|^3$ (D) 3|A|

Solution:

Option (B) is correct.
Explanation:
$|adj.\ A| = |A|^{n-1} = |A|^2$ (for n = 3)

18. If A is an invertible matrix of order 2, then det (A^{-1}) is equal to:

(A) det A (B) 1/ det A (C) 1 (D) 0

Solution:

Option (B) is correct.

Explanation:
$A\ A^{-1} = I$
$\det (A\ A^{-1} = I)$
$\det(A) \det(A^{-1}) = 1$

$\det(A^{-1}) = 1/\det A$

Exercise 4.6 Page No: 136

Examine the consistency of the system of equations in Exercises 1 to 6.

1. x +2y = 2: and 2x + 3y = 3

Solution:
Given set of equations is : x +2y = 2: and 2x + 3y = 3

This set of equation can be written in the form of matrix as AX = B, where

$A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and

$X = \begin{bmatrix} x \\ y \end{bmatrix}$

So, AX = B is

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = -1 \neq 0$$

Inverse of matrix exists. So system of equations is consistent.

2. 2x – y = 5 and x + y = 4

Solution:
Given set of equations is : 2x – y = 5 and x + y = 4

This set of equation can be written in the form of matrix as AX = B, where

$A = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$

So, AX = B is

$$\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} = 3 \neq 0$$

Inverse of matrix exists. So system of equations is consistent.

3. x + 3y and 2x + 6y = 8

Solution:
Given set of equations is : x + 3y and 2x + 6y = 8

This set of equation can be written in the form of matrix as AX = B.

$$\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$$

Where,

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \text{ and } B = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 3 \\ 2 & 6 \end{vmatrix} = 0$$

$$\text{adj. } A = \begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix}$$

And

$$(\text{adj. } A)B = \begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 8 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \end{bmatrix} \neq 0$$

The given equations are inconsistent.

4. x + y + z =1; 2x + 3y + 2z = 2 and ax + ay + 2az = 4

Solution:
Given set of equations is : x + y + z =1; 2x + 3y + 2z = 2 and ax + ay + 2az = 4

This set of equation can be written in the form of matrix as AX = B

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ a & a & 2a \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ a & a & 2a \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ a & a & 2a \end{vmatrix} = 1(6a-2a)-1(4a-2a)+1(2a-3a)$$

$$= 4a - 2a - a = a \neq 0$$

System of equations is consistent.

5. 3x – y – 2z = 2; 2y – z = -1 and 3x – 5y = 3

Solution:

Given set of equations is : 3x – y – 2z = 2; 2y – z = -1 and 3x – 5y = 3

This set of equation can be written in the form of matrix as AX = B

$$\begin{bmatrix} 3 & -1 & -2 \\ 0 & 2 & -1 \\ 3 & -5 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & -1 & -2 \\ 0 & 2 & -1 \\ 3 & -5 & 0 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 3 & -1 & -2 \\ 0 & 2 & -1 \\ 3 & -5 & 0 \end{vmatrix}$$

= 3(-5) +(3) -2(-6)

= 15 – 15

= 0

Now,

$$(\text{adj. } A) = \begin{bmatrix} -5 & 10 & 5 \\ -3 & 6 & 3 \\ -6 & 12 & 6 \end{bmatrix}$$

$$(\text{adj. }A)B = \begin{bmatrix} -5 & 10 & 5 \\ -3 & 6 & 3 \\ -6 & 12 & 6 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -5 \\ -3 \\ -6 \end{bmatrix} \neq 0$$

6. Given set of equations is :
5x – y + 4z = 5
2x + 3y + 5z = 2
5x – 2y + 6z = –1

Solution:
Given set of equations is: 5x – y + 4z = 5 ; 2x + 3y + 5z = 2; 5x – 2y + 6z = –1

This set of equation can be written in the form of matrix as AX = B

$$\begin{bmatrix} 5 & -1 & 4 \\ 2 & 3 & 5 \\ 5 & -2 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix}$$

$$A = \begin{bmatrix} 5 & -1 & 4 \\ 2 & 3 & 5 \\ 5 & -2 & 6 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 5 & -1 & 4 \\ 2 & 3 & 5 \\ 5 & -2 & 6 \end{vmatrix}$$

= 5(18 + 10) + 1(12-25) + 4(-4 – 15)

= 140 – 13 -76

= 140- 89

= 51

≠ 0

System of equations is consistent.

Solve system of linear equations, using matrix method, in Exercises 7 to 14.

7. 5x + 2y = 4 and 7x + 3y = 5
Solution:

Given set of equations is : 5x + 2y = 4 and 7x + 3y = 5

This set of equation can be written in the form of matrix as AX = B

$$\begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

Where,

$$A = \begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix}$$

And |A| = 1 ≠ 0
System is consistent.
Now,

$$X = A^{-1}B = \frac{1}{|A|}(\text{adj. } A)B$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{1}\begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix}\begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} 12-10 \\ -28+25 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

=> x = 2 and y = -3

8. 2x - y = -2 and 3x + 4y = 3

Solution:
Given set of equations is : 2x - y = -2 and 3x + 4y = 3
This set of equation can be written in the form of matrix as AX = B

$$\begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$$

|A| = 11 ≠ 0

System is consistent.
So,

$$X = A^{-1}B = \frac{1}{|A|}(\text{adj. } A)B$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} -2 \\ 3 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} -8+3 \\ 6+6 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} -5 \\ 12 \end{bmatrix}$$

Therefore, x = -5/11 and y = 12/11

9. 4x – 3y = 3 and 3x – 5y = 7

Solution:
Given set of equations is : 4x – 3y = 3 and 3x – 5y = 7
This set of equation can be written in the form of matrix as AX = B

$$\begin{bmatrix} 4 & -3 \\ 3 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$$

Where,

$$A = \begin{bmatrix} 4 & -3 \\ 3 & -5 \end{bmatrix}$$

And |A| = -20 + 9 = -11 ≠ 0
System is consistent.
So,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{-11}\begin{bmatrix} -5 & 3 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 3 \\ 7 \end{bmatrix}$$

$$= \frac{1}{-11}\begin{bmatrix} -15+21 \\ -9+28 \end{bmatrix} = \frac{1}{-11}\begin{bmatrix} 6 \\ 19 \end{bmatrix}$$

Therefore, x = 6/-11 and y = 19/-11

10. 5x + 2y = 3 and 3x + 2y = 5

Solution:
Given set of equations is : 5x + 2y = 3 and 3x + 2y = 5

This set of equation can be written in the form of matrix as AX = B

$$\begin{bmatrix} 5 & 2 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

Where

$$A = \begin{bmatrix} 5 & 2 \\ 3 & 2 \end{bmatrix}$$

And |A| = 4 ≠ 0

System is consistent.
So,

$$X = A^{-1}B = \frac{1}{|A|}(\text{adj. } A)B$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 2 & -2 \\ -3 & 5 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

Therefore, x = -1 and y = 4.

11. 2x + y + z = 1 and x – 2y – z = 3/2 and 3y – 5z = 9

Solution:
Given set of equations is : 2x + y + z = 1 and x – 2y – z = 3/2 and 3y – 5z = 9

This set of equation can be written in the form of matrix as AX = B

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & -2 & -1 \\ 0 & 3 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3/2 \\ 9 \end{bmatrix}$$

Where

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & -2 & -1 \\ 0 & 3 & -5 \end{bmatrix}$$

And

$$|A| = \begin{vmatrix} 2 & 1 & 1 \\ 1 & -2 & -1 \\ 0 & 3 & -5 \end{vmatrix}$$

$$= 34 \neq 0$$

System is consistent.
So,

$$X = A^{-1}B = \frac{1}{|A|}(\text{adj. } A)B$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{34}\begin{bmatrix} 13 & 8 & 1 \\ 5 & -10 & 3 \\ 3 & -6 & -5 \end{bmatrix}\begin{bmatrix} 1 \\ \frac{3}{2} \\ 9 \end{bmatrix}$$

$$= \frac{1}{34}\begin{bmatrix} 13+12+9 \\ 5-15+27 \\ 3-9-45 \end{bmatrix} = \frac{1}{34}\begin{bmatrix} 34 \\ 17 \\ -51 \end{bmatrix}$$

Therefore, x = 1, y = ½ and z = 3/2

12. x – y + z = 4 and 2x + y – 3z = 0 and x + y + z = 2

Solution:

Given set of equations is : x – y + z = 4 and 2x + y – 3z = 0 and x + y + z = 2
This set of equation can be written in the form of matrix as AX = B

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$$

Where,

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}$$

And

$$|A| = \begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{vmatrix}$$

= 10 ≠ 0
System is consistent.
So,

$$X = A^{-1}B = \frac{1}{|A|}(\text{adj. }A)B$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}\begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$$

$$= \frac{1}{10}\begin{bmatrix} 16+0+4 \\ -20+0+10 \\ 4-0+6 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

Therefore, x = 2, y = -1 and z = 1

13.
2x + 3y + 3z = 5
x – 2y + z = – 4
3x – y – 2z = 3
Solution:

Given set of equations is :
2x + 3y + 3z = 5
x – 2y + z = – 4
3x – y – 2z = 3

This set of equation can be written in the form of matrix as AX = B

$$\begin{bmatrix} 2 & 3 & 3 \\ 1 & -2 & 1 \\ 3 & -1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix}$$

Where

$$A = \begin{bmatrix} 2 & 3 & 3 \\ 1 & -2 & 1 \\ 3 & -1 & -2 \end{bmatrix}$$

And,

$$|A| = \begin{vmatrix} 2 & 3 & 3 \\ 1 & -2 & 1 \\ 3 & -1 & -2 \end{vmatrix}$$

$= 40 \neq 0$

System is consistent.
So,

$$X = A^{-1}B = \frac{1}{|A|}(\text{adj. } A)B$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{40}\begin{bmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{bmatrix}\begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix}$$

$$= \frac{1}{40}\begin{bmatrix} 25-12+27 \\ 25+52+3 \\ 25-44-21 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

Therefore, x = 1, y = 2 and z = -1.

14.
x – y + 2z = 7
3x + 4y – 5z = – 5
2x – y + 3z = 12

Solution:
Given set of equations is :
x – y + 2z = 7
3x + 4y – 5z = – 5
2x – y + 3z = 12

This set of equation can be written in the form of matrix as AX = B

$$\begin{bmatrix} 1 & -1 & 2 \\ 3 & 4 & -5 \\ 2 & -1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ -5 \\ 12 \end{bmatrix}$$

Where,

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 4 & -5 \\ 2 & -1 & 3 \end{bmatrix}$$

And

$$|A| = \begin{vmatrix} 1 & -1 & 2 \\ 3 & 4 & -5 \\ 2 & -1 & 3 \end{vmatrix}$$

$= 4 \neq 0$

System is consistent.
So,

$$X = A^{-1}B = \frac{1}{|A|}(\text{adj. } A)B$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 7 & 1 & -3 \\ -19 & -1 & 11 \\ -11 & -1 & 7 \end{bmatrix}\begin{bmatrix} 7 \\ -5 \\ 12 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 49-5-36 \\ -133+5+132 \\ -77+5+84 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

Therefore, x = 2, y = 1 and z = 3.

15. If A = $\begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$, find A^{-1}. Using A^{-1} solve the system of equations.
2x – 3y + 5z = 11
3x + 2y – 4z = – 5
x + y – 2z = – 3

Solution:

$$A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{vmatrix}$$

= -1 ≠ 0; Inverse of matrix exists.

Find the inverse of matrix:
Cofactors of matrix:

$$A_{11} = 0, A_{12} = 2, A_{13} = 1$$

$$A_{21} = -1, A_{22} = -9, A_{23} = -5$$

$$A_{31} = 2, A_{32} = 23, A_{33} = 13$$

$$\text{adj. } A = \begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix}$$

So,

$$A^{-1} = \frac{1}{-1}\begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix}$$

Now, matrix of equation can be written as:
AX = B

$$\begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}$$

And, $X = A^{-1} B$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix}\begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix}\begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Therefore, x = 1, y = 2 and z =3.

16. The cost of 4 kg onion, 3 kg wheat and 2 kg rice is Rs. 60. The cost of 2 kg onion, 4 kg wheat and 2 kg rice is Rs. 90. The cost of 6 kg onion, 2 k wheat and 3 kg rice is Rs. 70. Find cost of each item per kg by matrix method.

Solution:
Let x, y and z be the per kg. prices of onion, wheat and rice respectively.
According to given statement, we have following equations,
4x + 3y + 2z = 60
2x + 4y + 6z = 90
6x + 2y + 3z = 70

The above system of equations can be written in the form of matrix as, AX = B

$$\begin{bmatrix} 4 & 3 & 2 \\ 2 & 4 & 6 \\ 6 & 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 60 \\ 90 \\ 70 \end{bmatrix}$$

Where,

$$A = \begin{bmatrix} 4 & 3 & 2 \\ 2 & 4 & 6 \\ 6 & 2 & 3 \end{bmatrix}$$

And

$$|A| = \begin{vmatrix} 4 & 3 & 2 \\ 2 & 4 & 6 \\ 6 & 2 & 3 \end{vmatrix}$$

= 4(0) – 3(-30) + 2(-20)

= 50 ≠ 0

System is consistent, and X = A^{-1} B

First find invers of A.
Cofactors of all the elements of A are:

$$A_{11} = 0, A_{12} = 30, A_{13} = -20$$
$$A_{21} = -5, A_{22} = 0, A_{23} = 10$$
$$A_{31} = 10, A_{32} = -20, A_{33} = 10$$

$$\text{adj. } A = \begin{bmatrix} 0 & -5 & 10 \\ 30 & 0 & -20 \\ -20 & 10 & 10 \end{bmatrix}$$

Again,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{50}\begin{bmatrix} 0 & -5 & 10 \\ 30 & 0 & -20 \\ -20 & 10 & 10 \end{bmatrix}\begin{bmatrix} 60 \\ 90 \\ 70 \end{bmatrix}$$

$$= \frac{1}{50}\begin{bmatrix} -450+700 \\ 1800-1400 \\ -1200+900+700 \end{bmatrix}$$

$$= \frac{1}{50}\begin{bmatrix} 250 \\ 400 \\ 400 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \\ 8 \end{bmatrix}$$

Therefore, x = 5, y = 8 and z = 8.
The cost of onion, wheat and rice per kg are Rs. 5, Rs, 8 and Rs. 8 respectively.

Miscellaneous Examples

1. Prove that the determinant $\begin{vmatrix} x & \sin\theta & \cos\theta \\ -\sin\theta & -x & 1 \\ \cos\theta & 1 & x \end{vmatrix}$ **is independent of θ.**

Solution:

Let $\Delta = \begin{bmatrix} x & \sin\theta & \cos\theta \\ -\sin\theta & -x & 1 \\ \cos\theta & 1 & x \end{bmatrix}$

$$\Delta = x\begin{vmatrix} -x & 1 \\ 1 & x \end{vmatrix} - \sin\theta\begin{vmatrix} -\sin\theta & 1 \\ \cos\theta & x \end{vmatrix} + \cos\theta\begin{vmatrix} -\sin\theta & -x \\ \cos\theta & 1 \end{vmatrix}$$

$$= x\left(-x^2-1\right) - \sin\theta\left(-x\sin\theta - \cos\theta\right) + \cos\theta\left(-\sin\theta + x\cos\theta\right)$$

$$= -x^3 - x + x\left(\sin^2\theta + \cos^2\theta\right)$$

$$= -x^3$$

Which is independent of θ (Proved)

2. Without expanding the determinant, prove that

$$\begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix} = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$$

Solution:

Start with LHS:

$$\begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix}$$

Multiplying R1 by a R2 by b and R3 by c , we have

$$\begin{vmatrix} a^2 & a^3 & abc \\ b^2 & b^3 & abc \\ c^2 & c^3 & abc \end{vmatrix}$$

Taking out common elements

$$\frac{abc}{abc}\begin{vmatrix} a^2 & a^3 & 1 \\ b^2 & b^3 & 1 \\ c^2 & c^3 & 1 \end{vmatrix}$$

Interchanging C_1 and C_3

$$= \begin{vmatrix} a^2 & a^3 & 1 \\ b^2 & b^3 & 1 \\ c^2 & c^3 & 1 \end{vmatrix} = -\begin{vmatrix} 1 & a^3 & a^2 \\ 1 & b^3 & b^2 \\ 1 & c^3 & c^2 \end{vmatrix}$$

Interchanging C_2 and C_3

$$= \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$$

=RHS (proved)

3. Evaluate

$$\begin{vmatrix} \cos\alpha\cos\beta & \cos\alpha\sin\beta & -\sin\alpha \\ -\sin\beta & \cos\beta & 0 \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta & \cos\alpha \end{vmatrix}$$

Solution:

$$\begin{vmatrix} \cos\alpha\cos\beta & \cos\alpha\sin\beta & -\sin\alpha \\ -\sin\beta & \cos\beta & 0 \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta & \cos\alpha \end{vmatrix}$$

$$= \cos\alpha\cos\beta(\cos\alpha\cos\beta - 0) - \cos\alpha\sin\beta(-\cos\alpha\sin\beta - 0) - \sin\alpha\left(-\sin\alpha\sin^2\beta - \sin\alpha\cos^2\beta\right)$$

$$= \cos^2\alpha\cos^2\beta + \cos^2\alpha\sin^2\beta + \sin^2\alpha\left(\sin^2\beta + \cos^2\beta\right)$$

$$= \cos^2\alpha\left(\cos^2\beta + \sin^2\beta\right) + \sin^2\alpha\left(\sin^2\beta + \cos^2\beta\right)$$

$$= \cos^2\alpha + \sin^2\alpha = 1$$

4. If a, b and c are real numbers, and

$$\Delta = \begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 0$$

Show that either a + b + c = 0 or a = b = c

Solution:

$$\begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$$

Applying: $R_1 \to R_1 + R_2 + R_3$

$$\begin{vmatrix} 2(a+b+c) & 2(a+b+c) & 2(a+b+c) \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$$

$$2(a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$$

Since $\Delta = 0$
This implies,
Either 2(a + b+ c) = 0 or

$$\begin{vmatrix} 1 & 1 & 1 \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 0$$

Case 1: If 2(a + b+ c) = 0
Then (a + b+ c) = 0

Case 2:

$$\begin{vmatrix} 1 & 1 & 1 \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 0$$

Applying: $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$

$$\begin{vmatrix} 1 & 0 & 0 \\ c+a & a+b-c-a & b+c-c-a \\ a+b & b+c-a-b & c+a-a-b \end{vmatrix} = 0$$

or $\begin{vmatrix} b-c & b-a \\ c-a & c-b \end{vmatrix} = 0$

=>(b-c)(c-b) – (b-a)(c-a) = 0

=>$bc-b^2-c^2+bc-bc+ab+ac-a^2=0$

=>$-a^2-b^2-c^2+ab+bc+ca=0$

=> $a^2+a^2+b^2+b^2+c^2+c^2-2ab-2bc-2ca=0$

=>$\left(a^2+b^2-2ab\right)+\left(b^2+c^2-2bc\right)+\left(a^2+c^2-2ca\right)=0$

=>$(a-b)^2 + (b-c)^2 + (c-a)^2 = 0$
Above expression only possible, if (a – b) = 0 and (b – c) = 0 and (c – a) = 0

That is a = b and b = c and c = a
Therefore, we have result, either a+b+c = 0 or a=b=c.

5. Solve the equation

$$\Delta = \begin{vmatrix} x+a & x & x \\ x & x+a & x \\ x & x & x+a \end{vmatrix} = 0, a \neq 0$$

Solution:

$$\begin{vmatrix} x+a & x & x \\ x & x+a & x \\ x & x & x+a \end{vmatrix} = 0$$

Applying: $R_1 \to R_1 + R_2 + R_3$

$$\begin{vmatrix} 3x+a & 3x+a & 3x+a \\ x & x+a & x \\ x & x & x+a \end{vmatrix} = 0$$

$$(3x+a)\begin{vmatrix} 1 & 1 & 1 \\ x & x+a & x \\ x & x & x+a \end{vmatrix} = 0$$

Case 1: Either 3x + a = 0

then x = -a/3

Case 2: or

$$\begin{vmatrix} 1 & 1 & 1 \\ x & x+a & x \\ x & x & x+a \end{vmatrix} = 0$$

Applying:

$C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$

$$\begin{vmatrix} 1 & 0 & 0 \\ x & a & 0 \\ x & 0 & a \end{vmatrix} = 0$$

a^2 = 0

or a = 0

Not possible, as we are given a ≠ 0.

So , x = -a/3 is only the solution.

6. Prove that

$$\Delta = \begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix} = 4a^2b^2c^2$$

Solution:

LHS:

$$\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix}$$

Taking a, b, and c from all the row1, row 2 and row 3 respectively.

$$= abc\begin{vmatrix} a & c & (a+c) \\ (a+b) & b & a \\ b & (b+c) & c \end{vmatrix}$$

$R_1 \to R_1 - R_2 - R_3$

$$= abc\begin{vmatrix} a-a-b-b & c-b-b-c & a+c-a-c \\ a+b & b & a \\ b & b+c & c \end{vmatrix}$$

$C_2 \to C_2 - C_1$

$$= abc\begin{vmatrix} -2b & -2b & 0 \\ a+b & b & a \\ b & b+c & c \end{vmatrix} = abc\begin{vmatrix} -2b & 0 & 0 \\ a+b & -a & a \\ b & c & c \end{vmatrix}$$

$$= abc(-2b)(-ac-ac) = 4a^2b^2c^2$$

RHS (Proved)

7. If $A^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$ **and B = find $(AB)^{-1}$.**

Solution:

As we know, $(AB)^{-1} = B^{-1}A^{-1}$ …(1)

First find inverse of matrix B.

$$|B| = \begin{vmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{vmatrix}$$

$= 1(3-0) - 2(-1-0) + (-2)(2-0)$ = 1≠ 0 (Inverse of B is possible)

Find cofactors of B:

$B_{11}=3, B_{12}=1, B_{13}=2$
$B_{21}=2, B_{22}=1, B_{23}=2$ and
$B_{31}=6, B_{32}=2, B_{33}=5$

So adj. of B is

$$\begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$$

Now,

$$B^{-1}=\frac{1}{|B|}(\text{adj. } B)=\frac{1}{1}\begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$$

From equation (1),

$$(AB)^{-1}=\begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}\begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$$

=

$$\begin{bmatrix} 9 & -3 & 5 \\ -2 & 3 & 1 \\ 1 & 0 & 2 \end{bmatrix}$$

8. Let A = $\begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$**, verify that**

(i) $(\text{adj. } A)^{-1} = \text{adj. } (A^{-1})$

(ii) $(A^{-1})^{-1} = A$

Solution:

$$|A|=\begin{vmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{vmatrix}$$

= -13 ≠ 0 (Inverse of A exists)

Cofactors of A are:

$A_{11} = 14, A_{12} = 11, A_{13} = -5$

$A_{21} = 11, A_{22} = 4, A_{23} = -3$

$A_{31} = -5, A_{32} = -3, A_{33} = -1$

So, adjoint of A is

$$\begin{bmatrix} 14 & 11 & -5 \\ 11 & 4 & -3 \\ -5 & -3 & -1 \end{bmatrix}$$

Now, A^{-1}

$$\frac{-1}{13}\begin{bmatrix} 14 & 11 & -5 \\ 11 & 4 & -3 \\ -5 & -3 & -1 \end{bmatrix}$$

Again,

$$|B| = \begin{vmatrix} 14 & 11 & -5 \\ 11 & 4 & -3 \\ -5 & -3 & -1 \end{vmatrix}$$

= 169 ≠ 0 (Inverse of A exists)

Cofactors of B are:

$B_{11} = -13, B_{12} = 26, B_{13} = -13$

$B_{21} = 26, B_{22} = -39, B_{23} = -13$

$B_{31} = -13, B_{32} = -13, B_{33} = -65$

Therefore, Inverse of B is

$$\frac{-1}{13}\begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$$

Find: adj A^{-1}

$$A^{-1} = \frac{-1}{13}\begin{bmatrix} 14 & 11 & -5 \\ 11 & 4 & -3 \\ -5 & -3 & -1 \end{bmatrix}$$

$|A^{-1}| = -1/13 \neq 0$ (After solving the determinant we get the value. Try at your own)

Inverse of A^{-1} exists.

Let say cofactors of A^{-1} are represented as C_{ij} , we have

$$C_{11} = \frac{-1}{13}, C_{12} = \frac{2}{13}, C_{13} = \frac{-1}{13}$$
$$C_{21} = \frac{2}{13}, C_{22} = \frac{-3}{13}, C_{23} = \frac{-1}{13}$$ and

$$C_{31} = \frac{-1}{13}, C_{32} = \frac{-1}{13}, C_{33} = \frac{-5}{13}$$

Therefore:
$$\left(A^{-1}\right)^{-1} = C^{-1} = \frac{1}{|C|}(\text{adj. } C)$$

Which implies,

$$\begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$$
= A

Which is again given matrix A.

(i) $$(\text{adj. } A)^{-1} = \text{adj. } \left(A^{-1}\right)$$

$$\frac{-1}{13}\begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix} = \frac{-1}{13}\begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$$

(ii) $\left(A^{-1}\right)^{-1} = A$

$$\frac{-1}{13}\begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$$

9. Evaluate

$$\begin{vmatrix} x & y & x+y \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$$

Solution:
Consider,

$$\Delta = \begin{vmatrix} x & y & x+y \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$$

Operation: $[R_1 \to R_1 + R_2 + R_3]$

$$\begin{vmatrix} 2(x+y) & 2(x+y) & 2(x+y) \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$$

Taking 2(x + y) common from first row

$$2(x+y)\begin{vmatrix} 1 & 1 & 1 \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$$

Operation: $[C_2 \to C_2 - C_1 \text{ and } C_3 \to C_3 - C_1]$

$$2(x+y)\begin{vmatrix} 1 & 0 & 0 \\ y & x+y-y & x-y \\ x+y & x-x-y & y-x-y \end{vmatrix}$$

$$2(x+y).1\begin{vmatrix} x & x-y \\ -y & -x \end{vmatrix}$$

$$2(x+y)\{-x^2+y(x-y)\}$$

$$-2(x+y)(x^2-xy+y^2)$$

=-2(x^3 + y^3)

Δ =-2(x^3 + y^3)

10. Evaluate

$$\begin{vmatrix} 1 & x & y \\ 1 & x+y & y \\ 1 & x & x+y \end{vmatrix}$$

Solution:
Consider

$$\Delta = \begin{vmatrix} 1 & x & y \\ 1 & x+y & y \\ 1 & x & x+y \end{vmatrix}$$

Operation: $[R_2 \to R_2 - R_1 \text{ and } R_3 \to R_3 - R_1]$

$$\begin{vmatrix} 1 & x & y \\ 0 & x+y-x & 0 \\ 0 & 0 & x+y-y \end{vmatrix}$$

$$\begin{vmatrix} 1 & x & y \\ 0 & y & 0 \\ 0 & 0 & x \end{vmatrix}$$

=xy

⇨ Δ = xy

Using properties of determinants in Exercises 11 to 15, prove that:

11.

$$\begin{vmatrix} \alpha & \alpha^2 & \beta+\gamma \\ \beta & \beta^2 & \gamma+\alpha \\ \gamma & \gamma^2 & \alpha+\beta \end{vmatrix} = (\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)(\alpha+\beta+\gamma)$$

Solution:
LHS

$$\begin{vmatrix} \alpha & \alpha^2 & \beta+\gamma \\ \beta & \beta^2 & \gamma+\alpha \\ \gamma & \gamma^2 & \alpha+\beta \end{vmatrix}$$

Operation: $[C_3 \to C_3 + C_1]$

$$\begin{vmatrix} \alpha & \alpha^2 & \alpha+\beta+\gamma \\ \beta & \beta^2 & \alpha+\beta+\gamma \\ \gamma & \gamma^2 & \alpha+\beta+\gamma \end{vmatrix}$$

$$(\alpha+\beta+\gamma)\begin{vmatrix} \alpha & \alpha^2 & 1 \\ \beta & \beta^2 & 1 \\ \gamma & \gamma^2 & 1 \end{vmatrix}$$

Operation: $[R_2 \to R_2 - R_1 \text{ and } R_3 \to R_3 - R_1]$

$$(\alpha+\beta+\gamma)\begin{vmatrix} \alpha & \alpha^2 & 1 \\ \beta-\alpha & \beta^2-\alpha^2 & 0 \\ \gamma-\alpha & \gamma^2-\alpha^2 & 0 \end{vmatrix}$$

=

$$(\alpha+\beta+\gamma)\begin{vmatrix} \beta-\alpha & (\beta-\alpha)(\beta+\alpha) \\ \gamma-\alpha & (\gamma-\alpha)(\gamma+\alpha) \end{vmatrix}$$

=

$$(\alpha+\beta+\gamma)(\beta-\alpha)(\gamma-\alpha)\begin{vmatrix} 1 & (\beta+\alpha) \\ 1 & (\gamma+\alpha) \end{vmatrix}$$

= $(\alpha+\beta+\gamma)(\beta-\alpha)(\gamma-\alpha)(\gamma+\alpha-\beta-\alpha)$

= $(\alpha+\beta+\gamma)[-(\alpha-\beta)](\gamma-\alpha)[-(\beta-\gamma)]$

= $(\alpha+\beta+\gamma)(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)$

= RHS

12.

$$\begin{vmatrix} x & x^2 & 1+px^3 \\ y & y^2 & 1+py^3 \\ z & z^2 & 1+pz^3 \end{vmatrix} = (1+pxyz)(x-y)(y-z)(z-x)$$

Solution:

LHS= $\begin{vmatrix} x & x^2 & 1+px^3 \\ y & y^2 & 1+py^3 \\ z & z^2 & 1+pz^3 \end{vmatrix}$

= $\begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + \begin{vmatrix} x & x^2 & px^3 \\ y & y^2 & py^3 \\ z & z^2 & pz^3 \end{vmatrix}$

We have two determinants, say Δ_1 and Δ_2

= $\Delta_1 + \Delta_2$

$$\Delta_1 = \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix}$$

Operation: $[R_2 \to R_2 - R_1 \text{ and } R_3 \to R_3 - R_1]$

= $\begin{vmatrix} x & x^2 & 1 \\ y-x & y^2-x^2 & 0 \\ z-x & z^2-x^2 & 1 \end{vmatrix}$

= $\begin{vmatrix} y-x & (y-x)(y+x) \\ z-x & (z-x)(z+x) \end{vmatrix}$

= $(y-x)(z-x)\begin{vmatrix} 1 & y+x \\ 1 & z+x \end{vmatrix}$

= $(y-x)(z-x)(z+x-y-x)$

= $(x-y)(y-z)(z-x)$

Again:

$$\Delta_2 = \begin{vmatrix} x & x^2 & px^3 \\ y & y^2 & py^3 \\ z & z^2 & pz^3 \end{vmatrix}$$

$$pxyz\Delta_2 = \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$

= $-pxyz\begin{vmatrix} x^2 & x & 1 \\ y^2 & y & 1 \\ z^2 & z & 1 \end{vmatrix}$

= $pxyz\begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix}$

= pxyz Δ_1

Therefore: $\Delta_1 + \Delta_2$

LHS

= $(y-x)(z-x)(z-y)+pxyz(y-x)(z-x)(z-y)$

= $(1+pxyz)(y-x)(z-x)(z-y)$

= RHS
(Proved)

13. Prove that

$$\begin{vmatrix} 3a & -a+b & -a+c \\ -b+a & 3b & -b+c \\ -c+a & -c+b & 3c \end{vmatrix} = 3(a+b+c)(ab+bc+ca)$$

Solution:
LHS

$$\begin{vmatrix} 3a & -a+b & -a+c \\ -b+a & 3b & -b+c \\ -c+a & -c+b & 3c \end{vmatrix}$$

Operation: $[C_1 \rightarrow C_1 + C_2 + C_3]$

$$\begin{vmatrix} a+b+c & -a+b & -a+c \\ a+b+c & 3b & -b+c \\ a+b+c & -c+b & 3c \end{vmatrix}$$

$$(a+b+c)\begin{vmatrix} 1 & -a+b & -a+c \\ 1 & 3b & -b+c \\ 1 & -c+b & 3c \end{vmatrix}$$

Operation: $[R_2 \rightarrow R_2 - R_1 \text{ and } R_3 \rightarrow R_3 - R_1]$

$$(a+b+c).1\begin{vmatrix} 2b+a & a-b \\ a-c & 2c+a \end{vmatrix}$$

= $(a+b+c)[(2b+a)(2c+a)-(a-b)(a-c)]$

= $(a+b+c)[4bc+2ab+a^2-a^2+ac+ab-bc]$

= $3(a+b+c)(ab+bc+ac)$

= RHS

Hence Proved.

14. Prove that

$$\begin{vmatrix} 1 & 1+p & 1+p+q \\ 2 & 3+2p & 4+3p+2q \\ 3 & 6+3p & 10+6p+3q \end{vmatrix} = 1$$

Solution:
LHS

$$\begin{vmatrix} 1 & 1+p & 1+p+q \\ 2 & 3+2p & 4+3p+2q \\ 3 & 6+3p & 10+6p+3q \end{vmatrix}$$

Operation: $[R_2 \to R_2 - 2R_1 \text{ and } R_3 \to R_3 - 3R_1]$

$$\begin{vmatrix} 1 & 1+p & 1+p+q \\ 0 & 1 & 2+p \\ 0 & 3 & 7+3p \end{vmatrix}$$

=

$$1\begin{vmatrix} 1 & 2+p \\ 3 & 7+3p \end{vmatrix} - 0 + 0$$

= $7+3p-6-3p$

= 1

=RHS

Hence Proved.

15. Prove that

$$\begin{vmatrix} \sin\alpha & \cos\alpha & \cos(\alpha+\delta) \\ \sin\beta & \cos\beta & \cos(\beta+\delta) \\ \sin\gamma & \cos\gamma & \cos(\gamma+\delta) \end{vmatrix} = 0$$

Solution:

LHS

$$\begin{vmatrix} \sin\alpha & \cos\alpha & \cos(\alpha+\delta) \\ \sin\beta & \cos\beta & \cos(\beta+\delta) \\ \sin\gamma & \cos\gamma & \cos(\gamma+\delta) \end{vmatrix}$$

=

$$\begin{vmatrix} \sin\alpha & \cos\alpha & \cos\alpha\cos\delta - \sin\alpha\sin\delta \\ \sin\beta & \cos\beta & \cos\beta\cos\delta - \sin\beta\sin\delta \\ \sin\gamma & \cos\gamma & \cos\gamma\cos\delta - \sin\gamma\sin\delta \end{vmatrix}$$

Operation: $\left[C_3 \to C_3 + (\sin\delta)C_1\right]$

=

$$\begin{vmatrix} \sin\alpha & \cos\alpha & \cos\alpha\cos\delta - \sin\alpha\sin\delta + \sin\alpha\sin\delta \\ \sin\beta & \cos\beta & \cos\beta\cos\delta - \sin\beta\sin\delta + \sin\beta\sin\delta \\ \sin\gamma & \cos\gamma & \cos\gamma\cos\delta - \sin\gamma\sin\delta + \sin\gamma\sin\delta \end{vmatrix}$$

= $$\begin{vmatrix} \sin\alpha & \cos\alpha & \cos\alpha\cos\delta \\ \sin\beta & \cos\beta & \cos\beta\cos\delta \\ \sin\gamma & \cos\gamma & \cos\gamma\cos\delta \end{vmatrix}$$

= $$\cos\delta\begin{vmatrix} \sin\alpha & \cos\alpha & \cos\alpha \\ \sin\beta & \cos\beta & \cos\beta \\ \sin\gamma & \cos\gamma & \cos\gamma \end{vmatrix}$$

Column 2 and column 3 are identical, as per determinant property, value is zero.

= 0

= RHS

16. Solve the system of equations

$\frac{2}{x}+\frac{3}{y}+\frac{10}{z}=4; \quad \frac{4}{x}-\frac{6}{y}+\frac{5}{z}=1; \quad \frac{6}{x}+\frac{9}{y}-\frac{20}{z}=2$

Solution:

Let

$\frac{1}{x}=u, \frac{1}{y}=v$, $\frac{1}{z}=w$

We have

$2u+3v+10w=4;$

$4u-6v+5w=1;$ and

$6u+9v-20w=2$

Below is the matrix from the given equations: AX = B

$$\begin{bmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix}=\begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$$

Let say

$$A=\begin{bmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{bmatrix}$$

Then,

|A| = 1200 ≠ 0

A^{-1} exists.

Cofactors of A are:

$A_{11}=75, A_{12}=110, A_{13}=72$

$A_{21}=150, A_{22}=-100, A_{23}=0$ and

$A_{31}=75, A_{32}=30, A_{33}=-24$

adj. A = $\begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}$

Inverse of A is

$$A^{-1} = \frac{\text{adj.A}}{|A|} = \frac{1}{1200}\begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}$$

Resubstitute the values, to get answer in the form of x, y and z.

Since AX = B
X = A^{-1} B

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \frac{1}{1200}\begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \frac{1}{1200}\begin{bmatrix} 300+150+150 \\ 440-100+60 \\ 288+0-48 \end{bmatrix}$$

=

$$\frac{1}{1200}\begin{bmatrix} 600 \\ 400 \\ 240 \end{bmatrix}$$

Which means:
u = ½, v = 1/3 and w = 1/5

This implies:
x = 1/u = 2
y = 1/v = 3
z = 1/w = 5
Answer!

Choose the correct answer in Exercise 17 to 19.

17. If a, b , c are in A.P., then the determinant is $\begin{vmatrix} x+2 & x+3 & x+2a \\ x+3 & x+4 & x+2b \\ x+4 & x+5 & x+2c \end{vmatrix}$**:**

(A) 0 **(B) 1** **(C) x** **(D) 2x**

Solution:

Option (A) is correct.

Explanation:

Since a, b , c are in A.P.
So, b – a = c – b

Let

$$\Delta = \begin{vmatrix} x+2 & x+3 & x+2a \\ x+3 & x+4 & x+2b \\ x+4 & x+5 & x+2c \end{vmatrix}$$

Operation: $[R_2 \to R_2 - R_1 \text{ and } R_3 \to R_3 - R_2]$

$$\begin{vmatrix} x+2 & x+3 & x+2a \\ 1 & 1 & 2(b-a) \\ 1 & 1 & 2(c-b) \end{vmatrix}$$

$$\begin{vmatrix} x+2 & x+3 & x+2a \\ 1 & 1 & 2(b-a) \\ 1 & 1 & 2(b-a) \end{vmatrix}$$

Row 2 and row 3 are identical, so value is zero.

= 0

18. If x, y, z are non-zero real numbers, then the inverse of matrix A = $\begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$ is:

(A)
$$\begin{bmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{1} \end{bmatrix}$$

(B)
$$xyz\begin{bmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{1} \end{bmatrix}$$

(C)
$$\frac{1}{xyz}\begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$$

(D)
$$\frac{1}{xyz}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution: Option (A) is correct.

Explanation:

Let A = $\begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$

|A| = xyz ≠ 0 ; (A^{-1} exists)

Now: Cofactors of A are:

$A_{11} = yz, A_{12} = 0, A_{13} = 0$
$A_{21} = 0, A_{22} = xz, A_{23} = 0$ and
$A_{31} = 0, A_{32} = 0, A_{33} = xy$

Therefore:

$$A^{-1} = \frac{\text{adj.A}}{|A|} = \frac{1}{xyz}\begin{vmatrix} yz & 0 & 0 \\ 0 & xz & 0 \\ 0 & 0 & xy \end{vmatrix}$$

$$= \begin{vmatrix} \frac{yz}{xyz} & 0 & 0 \\ 0 & \frac{xz}{xyz} & 0 \\ 0 & 0 & \frac{xy}{xyz} \end{vmatrix}$$

$$= \begin{vmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{-1} \end{vmatrix}$$

19. Let A = $\begin{bmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{bmatrix}$, **where** $0 \le \theta \le 2\pi$. **then:**

(A) Det (A) = 0

(B) Det (A) $\in (2, \infty)$

(C) Det (A) $\in (2, 4)$

(D) Det (A) $\in [2, 4]$

Solution:

Option (D) is correct.

Explanation:

Let A = $\begin{bmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{bmatrix}$

|A| = 2 + 2 sin^2 θ ≠ 0 ; (A^{-1} exists)

Since :

$-1 \le \sin\theta \le 1$

$0 \le \sin^2\theta \le 1$

(The value of θ cannot be negative)

So, $0 \le 2\sin^2\theta \le 2$

Add 2 in all the expressions:

$2 \le 2 + 2\sin^2\theta \le 4$

Which is equal to

$2 \le \text{Det. A} \le 4$